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1990 J. Phys.: Condens. Matter 2 405

(http://iopscience.iop.org/0953-8984/2/2/016)

J. Phys.: Condens. Matter 2 (1990) 405–415. Printed in the UK

Exact results on the dimerisation transition in SU(n) antiferromagnetic chains

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Received 16 May 1989

Abstract. We show that SU(n) antiferromagnetic chains (or equivalently SU(2) spin-s chains with Hamiltonians which project out singlet states) are exactly equivalent to the n^2-state quantum Potts chain (obtained from the transfer matrix for the 2D classical Potts model), for arbitrary n (or s with $n = 2s + 1$). This implies that the models are spontaneously dimerised with a finite gap for $n < 2$ but have a unique ground state and vanishing gap for $n \leqslant 2$. The $n \rightarrow 0$ limit may have some bearing on the quantum-Hall-effect localisation transition.

1. Introduction

The ordinary $s = \frac{1}{2}$ Heisenberg Hamiltonian

$$H = \tfrac{1}{4}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$$

can be written as a permutation operator:

$$(H + \tfrac{1}{8})|^{\alpha\beta}\rangle = \tfrac{1}{2}|^{\beta\alpha}\rangle$$

where α, β take on the values 1 and 2 and label the S^z eigen-states on two neighbouring sites. Alternatively, we may make a 'particle–hole transformation', defining

$$|_{\alpha}\rangle \equiv \varepsilon_{\alpha\beta}|^{\beta}\rangle$$

on the even sub-lattice. (Here repeated indices are summed from 1 to 2. $\varepsilon_{\alpha\beta}$ is the antisymmetric tensor with $\varepsilon_{12} = 1$.) Shifting H by a constant, it then becomes an annihilation operator:

$$H|^{\alpha}{}_{\beta}\rangle = -\tfrac{1}{2}\delta^{\alpha}{}_{\beta}|^{\gamma}{}_{\gamma}\rangle.$$

($\delta^{\alpha}{}_{\beta}$ is the Kronecker δ function.) The meaning of this becomes transparent if we represent the Heisenberg operators the way they arise in nature: in terms of electron operators. Letting $\psi_{i\alpha}$ annihilate an electron of spin α on the ith site, the original form of H is

$$H + \tfrac{1}{8}\mathbf{1} = \tfrac{1}{2}\psi^{+}{}_{1}{}^{\alpha}\psi_{1\beta}\psi^{+}{}_{2}{}^{\beta}\psi_{2\alpha}$$

and the second form is obtain by the particle–hole transformation, $\psi_{2\alpha} \rightarrow \psi^{+}{}_{2\alpha}$, so that

$$H = -\tfrac{1}{2}\psi^{+}{}_{1}{}^{\alpha}\psi_{1\beta}\psi^{+}{}_{2\alpha}\psi_{2}{}^{\beta} + \text{constant}.$$

In the first form H permutes a pair of electrons. In the second it annihilates a particle–

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Figure 1. The two valence-bond states for four sites.

hole pair with the same spin and creates a new pair with the same spin. These electron Hamiltonians are equivalent to the $s = \frac{1}{2}$ Heisenberg Hamiltonian if we restrict the number of electrons to be one on each site (or equivalently one hole on the even site.)

There are obvious generalisations with SU(n) symmetry for H written in either form: we simply let the indices run over n values rather than 2. The two generalisations are inequivalent for $n > 2$ because in the particle–particle case we have one particle on the even site and in the particle–hole case we have one hole or $n - 1$ particles. The representation of SU(n) on the odd site is the fundamental representation in both cases; that on the even site is the fundamental or anti-fundamental for the particle or hole case.

Both generalisations can be thought of as ordinary SU(2) spin-s antiferromagnets, where $n = 2s + 1$, the number of spin components, but with special Hamiltonians of higher (SU(n)) symmetry. The nature of these Hamiltonians is most easily understood by writing H in terms of projection operators onto the total spin on the pair of sites. In the particle–hole case H becomes:

$$H = -P^0(\boldsymbol{S}_1 + \boldsymbol{S}_2)$$

i.e. the energy is (-1) if the two spins are in a singlet state and is zero for all other possible spins, $1, 2, \ldots, 2s$. Note that this Hamiltonian has SU(n) symmetry because the n different S^z eigen-states are all treated equally by H. Explicitly, H can be written

$$H = -\prod_{j=1}^{2s} \{[-(\boldsymbol{S}_1 + \boldsymbol{S}_2)^2 + j(j+1)]/j(j+1)\}.$$

For $n = 3$ it is simply the pure biquadratic model,

$$H = -(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2)^2 + \text{constant}.$$

The particle–particle generalisation was solved by the Bethe *ansatz* [1] in the one-dimensional case; it has vanishing gap and power-law correlations like the $s = \frac{1}{2}$ case. The particle–hole generalisation was also shown to be Bethe-*ansatz* integrable [1]. It was shown to have a simple spectrum in the large-n limit in reference [2].

To see this it is convenient to introduce the valence-bond basis [2]. An arbitrary SU(n) singlet state is obtained by pair-wise contracting upper and lower indices. For instance, for four sites there are two possible contractions:

$$|^{\alpha}{}_{\alpha}{}^{\beta}{}_{\beta}\rangle \qquad |^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha}\rangle.$$

(Repeated lower case Greek indices are summed from 1 to n throughout this paper.) These are represented by diagrams with lines connecting the contracted sites, as in figure 1. The valence-bond states form a complete basis of SU(n) singlets.

(This follows from the fact that the only two invariant tensors for SU(n) are the Kronecker delta function, $\delta^{\alpha}{}_{\beta}$, and the Levi–Civita antisymmetric tensor: $\varepsilon^{\alpha_1\alpha_2\cdots\alpha_n}$. Contractions involving the ε tensor can always be rewritten in terms of the deltas using:

$$\varepsilon^{\alpha_1\alpha_2\cdots\alpha_n}\varepsilon_{\beta_1\beta_2\beta_3\ldots\beta_n} = \sum_{(s_1,s_2,s_3,\ldots,s_n)} \operatorname{sgn}(s_1, s_2, s_3, \ldots, s_n)\delta^{\alpha_1}{}_{\beta_{s_1}}\delta^{\alpha_2}{}_{\beta_{s_2}}\delta^{\alpha_3}{}_{\beta_{s_3}}\ldots\delta^{\alpha_n}{}_{\beta_{s_n}}$$

where the sum is over all permutations.)

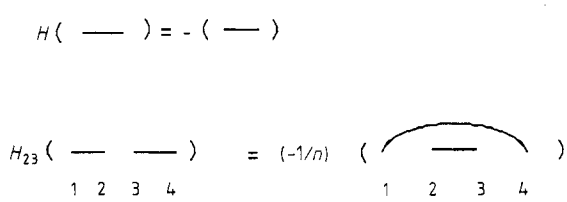

Figure 2. Diagrammatic definition of antiferromagnetic Hamiltonian on the valence-bond basis.

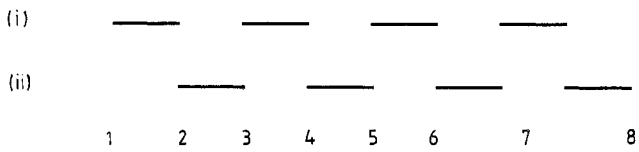

Figure 3. The two ground states in the large-n limit.

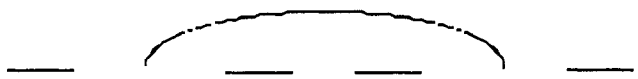

Figure 4. A valence-bond state with a longer bond. Note that there is one less nearest-neighbour valence bond than in figure 3.

The action of H on the valence-bond basis can be written in a simple diagrammatic way. It is convenient to normalise H so that

$$H|^{\alpha}{}_{\beta}\rangle = (-1/n)\delta^{\alpha}{}_{\beta}|^{\gamma}{}_{\gamma}\rangle.$$

Thus a link containing a valence bond is an eigen-state with eigenvalue -1:

$$H|^{\alpha}{}_{\alpha}\rangle = -|^{\alpha}{}_{\alpha}\rangle.$$

Now let us consider the action of H on a link not containing a valence bond. For instance, the action of H_{23}, the term coupling the second and third sites, on the state with 12 and 34 valence bonds is:

$$H_{23}|^{\alpha}{}_{\alpha}{}^{\beta}{}_{\beta}\rangle = -(1/n)|^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha}\rangle.$$

We obtain the state with 14 and 23 valence bonds, with amplitude $1/n$. These two simple rules are represented diagrammatically in figure 2. They completely define the action of H on *any* valence-bond state since sites 1 and 4 in the above equation need not be nearest neighbours of sites 2 and 3. We see that as n increases the amplitude for valence bonds to 'resonate' goes to zero like $1/n$. The reason for this is that a neighbouring particle and hole in different valence bonds will have the same index (and so be able to annihilate each other) only $1/n$ of the time. It is easy to see that, at large n, the two ground states will be approximately the two nearest-neighbour dimer configurations, drawn in figure 3. These states have energy $-L$, for a chain of $2L$ sites with periodic boundary conditions. All off-diagonal matrix elements of H are $O(1/n)$. All other singlet states are missing at least one nearest-neighbour valence bond (as in figure 4). Thus their energy is larger by at least $1 + O(1/n)$. Likewise, non-SU(n) singlets have some uncontracted indices and also have energies larger by $1 + O(1/n)$ (as in figure 5, in which broken lines denote uncontracted indices). Thus at large n the symmetry of translation by one site is spontaneously broken, and there is a finite gap. On the other hand, when $n = 2$ the ground

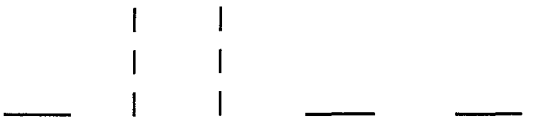

Figure 5. A non-singlet state with uncontracted indices represented by broken lines. Note that there is one less nearest-neighbour valence bond than in figure 3.

state is known to have full translational symmetry and zero gap from the Bethe *ansatz*. There must be a dimerisation transition at some critical value of n, $n_c \geqslant 2$. There is no *a priori* reason why n_c must be an integer; we can equally well define the model on the valence-bond states for any real n. A numerical investigation of this question was made in reference [3] for chains of length up to 26. There was clear evidence of a phase transition with a value of n_c around two or three.

The $n = 3$ case, corresponding to the pure biquadratic spin-1 chain, has been the subject of fairly extensive discussion in the context of the general bilinear–biquadratic chain:

$$H = \sum_i (\cos\theta \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \sin\theta (\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2).$$

The physical case, $\theta = 0$, has been argued to have a gap and a unique ground state [4]. The $\theta = \tan^{-1}(1/3)$ case was solved exactly (in terms of 'double valence bonds') and proven to have this property [5]. The $\theta = -\pi/4$ model is solvable using the Bethe *ansatz* and has vanishing gap and a unique ground state [6]. It was conjectured [7] that $\theta = -\pi/4$ represents a critical point separating undimerised ($\theta > -\pi/4$) and dimerised ($-\theta < \pi/4$) phases with gaps. This was supported by simple variational ground states of the two types [5]. A critical theory of the phase transition, involving the $k = 2$ Wess–Zumino–Witten non-linear σ model was developed [7], and shown [8] to give exact agreement with critical indices extracted from the Bethe *ansatz* solution [6] at $\theta = -\pi/4$. Numerical investigations of this conjecture [9, 3] have tended to show the gap first vanishing near the point $-\pi/4$ as it is approached from above, and dimerisation commencing at about this point, but with a zero or very small gap in the dimerised region right up to $\theta = -\pi/2$, the pure biquadratic model.

A breakthrough in the $n = 3$ case was obtained recently, building on earlier work of Parkinson [10], by Barber and Batchelor [11]. They argued that this model is exactly equivalent to the 9-state quantum Potts chain and hence has a spontaneously dimerised ground state. They were furthermore able to obtain exact results on the spectrum using the mapping of the Potts chain onto the $s = \frac{1}{2}$ xxz spin chain. (Their results also imply the integrability of the model, but this was already established in reference [1].) As will be shown in § 3, this mapping holds for all n, the SU(n) antiferromagnet being equivalent to the n^2-state Potts model. In the next section we will discuss how this mapping works in qualitative terms and what it implies for the antiferromagnet. In § 3 we give a partial proof of the mapping. We first show that the two problems both obey the same Temperley–Lieb algebra [12] with operators of the same dimension. We then show the mapping is exact for four sites. Finally we show it is exact for an arbitrary number of sites in the SU(n) singlet sector.

2. Mapping onto the Potts model

Following reference [11], where the $n = 3$ case was considered, we consider the more general SU(n) model with alternating interaction strength and *free* boundary conditions on a chain of length $2L$:

$$H = -(1/n)\sum_{\pm}\sum_{i=1}^{L-1} \varepsilon_{\pm}\psi^{+}{}_{2i\pm1}{}^{\alpha}\psi_{2i\pm1,\beta}\psi^{+}{}_{2i,\alpha}\psi_{2i}{}^{\beta} + \psi^{+}{}_{2L-1}{}^{\alpha}\psi_{2L-1,\beta}\psi^{+}{}_{2L,\alpha}\psi_{2L}{}^{\beta}$$

where $\varepsilon_{-} = 1$ and $\varepsilon_{+} = \lambda$.

The q-state Potts model has a classical Hamiltonian with a nearest-neighbour interaction between 'spins' taking on q values $s = 1, 2, 3, \ldots, q$ of the form:

$$H = -\sum_{\langle i,j\rangle} \delta_{s_i s_{i+1}}.$$

From the transfer matrix of the two-dimensional Potts model one can obtain [13] a corresponding one-dimensional quantum Hamiltonian:

$$H = -(1/q)\sum_{i=1}^{L}\sum_{k=0}^{q-1}(\Omega_i)^k - (\lambda/q)\sum_{i=1}^{L-1}\sum_{k=0}^{q-1}(R_i)^k(R_{i+1})^{q-k}.$$

(In [11] the summation index k runs from 1 to $(q-1)$ only. This simply corresponds to shifting H by a constant $(1+\lambda)/q$. We have also divided H by a factor of q relative to the normalisation in [11].) If we work in the classical Potts basis, we diagonalise the R_i:

$$R = \text{diag}\,(\exp(\mathrm{i}2\pi/q), \exp(\mathrm{i}4\pi/q), \exp(\mathrm{i}6\pi/q), \ldots, \exp(\mathrm{i}q2\pi/q)).$$

In this basis the second term in H becomes

$$H_{\mathrm{P}} = -\lambda\sum_{i=1}^{L-1}\delta_{s_i s_{i+1}}.$$

The s_i are the 'Potts spin' indices (labelling eigen-states of the R_i) and $\delta_{s_1 s_2}$ is the Kronecker delta function. Ω then looks like a transverse field, $\Omega = M$, which shifts the Potts spin index by one:

$$M_{ss'} = \delta_{s(s'+1)}.$$

(We identify $s+q$ with s.) We will refer to the first term as H_{f}, and the second as H_{P}. We may alternatively diagonalise the first term:

$$H_{\mathrm{f}} = -(1/q)\sum_{k=0}^{q-1}(\Omega_i)^k \rightarrow \text{diag}(-1, 0, 0, \ldots, 0).$$

The Potts term then is non-diagonal with $R = M$, i.e. the Potts spin indices on neighbouring sites are shifted in opposite directions by k, with $k = 0, 1, 2, \ldots, (q-1)$.

The Potts model is known to have an ordered phase for $\lambda > 1$ (corresponding to $T < T_{\mathrm{c}}$ in the classical problem). At large λ the Potts spins are essentially equal on all sites (up to small quantum fluctuations). For $\lambda < 1$ the system is disordered. For very small λ each site is essentially decoupled and will be in the state of eigenvalue -1, which is the symmetric sum over the q Potts spin states. This phase transition is known to be second order for $q \leqslant 4$ and first order for $q > 4$ [14].

The connection with the antiferromagnet is easy to understand for $L = 1$. The antiferromagnet has two sites and n^2 states. One is the SU(n) singlet, $|^{\alpha}{}_{\alpha}\rangle$, energy -1. The other $n^2 - 1$ are the adjoint representation of SU(n): $T_{\alpha}{}^{\beta}|^{\alpha}{}_{\beta}\rangle$, where the matrix T is traceless. These all have energy 0. Labelling the H_{f} eigen-states by a capital Greek index, $|\Lambda\rangle$, where Λ runs from 0 to $q-1$, we see that $|0\rangle$ corresponds to the SU(n) singlet, and $|A\rangle$, for $A = 1, 2, \ldots, q-1$, correspond to the adjoint representation states. (We will

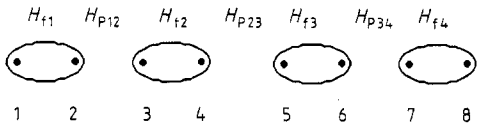

Figure 6. The identification of the Potts and field terms in the quantum Potts Hamiltonian with the links in the antiferromagnetic Hamiltonian.

let Arabic letters generally run over this restricted range.) For a longer Potts chain of length L, the field terms correspond to the antiferromagnet couplings with the odd site on the left:

$$H_{fi} \Leftrightarrow H_{2i-1,2i}$$

and the Potts terms correspond to the other half of the antiferromagnetic couplings:

$$H_{Pi,i+1} \Leftrightarrow H_{2i,2i+1}.$$

(see figure 6). Thus we see that the disordered phase where H_f is diagonalised separately on each site must correspond to the simple dimer state (figure 2, state a) with valence bonds between sites $2i-1$ and $2i$. On the other hand (and less obviously) the ordered state, where we diagonalise all the Potts terms must correspond to the other nearest-neighbour dimer state (figure 2, state b). There is a duality transformation in the Potts model which maps the ordered and disordered states into each other at $\lambda = 1$:

$$(R_{i+1/2'})^k \equiv \prod_{j \leqslant i} (\Omega_j)^{q-k} \qquad (\Omega_{i+1/2'})^k = R_i{}^k R_{i+1}{}^{(q-k)}.$$

Likewise, the two dimerisations of the antiferromagnet are equivalent. (Actually, both these statements are only true in the infinite length limit, for chains with free boundaries.) Since the Potts transition is first order at $q > 4$, the ordered and disordered ground states coexist as distinct degenerate states at $\lambda = 1$ in this case. This means that the SU(n) chain has two ground states at $\lambda = 1$ for $n > 2$; i.e. it is spontaneously dimerised. On the other hand, since the Potts transition is second order for $q \leqslant 4$, the two ground states become the same at $\lambda = 1$, so the SU(n) system is undimerised for $n \leqslant 2$. Furthermore, eigenstates of the Potts Hamiltonian can be obtained from an exact mapping [12] onto the $s = \frac{1}{2}$ xxz model; $q > 4$ ($q < 4$) corresponding to Ising (xy) anisotropy. Bethe *ansatz* results on the excitation spectrum [15] together with well established field-theory results on the correlation functions [16] imply that the model has a finite gap and correlation length for $q > 4$ but vanishing gap and power-law correlations for $q \leqslant 4$. Thus we conclude that $n_c = 2$ for the SU(n) antiferromagnet and the conjecture [7] about the gap in the dimerised phase was correct for the $n = 3$ biquadratic spin-1 model. The exact results on the Potts spectrum show [11] that the gap is in fact very small for $n = 3$, about 0.06, explaining the difficulty with the numerical work.

This result represents, to our knowledge, the first exact solution of an SU(n) invariant antiferromagnet for fractional n. As such, it may be a step towards finding the exact critical exponents for the quantum-Hall-effect localisation transition. It has been argued that the critical theory for this transition is a $(1+1)$-dimensional non-linear σ model with SU(n) symmetry and topological angle $\theta = \pi$, in the limit $n \to 0$ [17]. A programme to solve for the exact exponents was initiated in [18] by obtaining the σ models with $\theta = \pi$ from associated SU(n) antiferromagnets (generalising the correspondence between ordinary SU(2) antiferromagnets and the O(3) non-linear σ model [4]. However, it was realised in [19] that the σ models are in a massive phase at $\theta = \pi$ for $n \geqslant 3$; the only

massless case for physical values of n is $n = 2$. Thus the extrapolation of critical exponents to $n = 0$ requires the definition (and solution) of the models for fractional n. Unfortunately, although the antiferromagnetic models discussed here have the right qualitative behaviour (critical for $n \leqslant 2$), they involve SU(n) variables transforming under the wrong representation to obtain the non-linear σ models. The required representations have Young tableaux with $n/2$ columns and N rows where $N \to \infty$ (and n is even) [18]. Nevertheless, it is encouraging that the critical theory has been solved for all n in the simpler case of the fundamental representation and behaviour of the required type is obtained.

3. A partial proof of equivalence

3.1. Temperley–Lieb algebra

The terms in the antiferromagnetic Hamiltonian obey the same Temperley–Lieb algebra [12] as do the terms in the Potts Hamiltonian. (For the $n = 3$ case this was shown in [11].) This is easily shown using the fermion representation:

$$H_{12} = -\psi^{+}{}_{1}{}^{\alpha}\psi_{1\beta}\psi^{+}{}_{2\alpha}\psi_{2}{}^{\beta}$$

(We drop the $1/n$ normalisation factor in our definition of H for this sub-section.) Using the constraint of one particle (hole) per site we have

$$\psi^{+}{}_{1}{}^{\alpha}\psi_{1\beta}\psi^{+}{}_{1}{}^{\gamma}\psi_{1\delta} = \delta^{\gamma}{}_{\beta}\psi^{+}{}_{1}{}^{\alpha}\psi_{1\delta}.$$

Thus we obtain:

$$H_{12}{}^{2} = (\psi^{+}{}_{1}{}^{\alpha}\psi_{1\beta}\psi^{+}{}_{1}{}^{\gamma}\psi_{1\delta})(\psi^{+}{}_{2\alpha}\psi_{2}{}^{\beta}\psi^{+}{}_{2\gamma}\psi_{2}{}^{\delta}) = (\delta^{\gamma}{}_{\beta}\psi^{+}{}_{1}{}^{\alpha}\psi_{1\delta})(\delta^{\beta}{}_{\gamma}\psi^{+}{}_{2\alpha}\psi_{2}{}^{\delta})$$
$$= n\psi^{+}{}_{1}{}^{\alpha}\psi_{1\delta}\psi^{+}{}_{2\alpha}\psi_{2}{}^{\delta} = nH_{12}.$$

Likewise:

$$H_{12}H_{23}H_{12} = (\psi^{+}{}_{1}{}^{\alpha}\psi_{1\beta}\psi^{+}{}_{1}{}^{\varepsilon}\psi_{1\nu})(\psi^{+}{}_{2\alpha}\psi_{2}{}^{\beta}\psi^{+}{}_{2\gamma}\psi_{2}{}^{\delta}\psi^{+}{}_{2\varepsilon}\psi_{2}{}^{\nu})$$
$$(\psi^{+}{}_{3}{}^{\gamma}\psi_{3\delta}) = (\delta_{\beta}{}^{\varepsilon}\psi^{+}{}_{1}{}^{\alpha}\psi_{1\nu})(\delta^{\delta}{}_{\varepsilon}\delta^{\beta}{}_{\gamma}\psi^{+}{}_{2\alpha}\psi_{2}{}^{\nu})(\psi^{+}{}_{3}{}^{\gamma}\psi_{3\delta})$$
$$= (\psi^{+\alpha}{}_{1}\psi_{1\nu})(\psi^{+}{}_{2\alpha}\psi_{2}{}^{\nu})(\psi^{+}{}_{3}{}^{\gamma}\psi_{3\gamma}) = (\psi^{+}{}_{1}{}^{\alpha}\psi_{1\nu})(\psi^{+}{}_{2\alpha}\psi_{2}{}^{\nu}) = H_{12}.$$

Together with $[H_{12}, H_{34}] = 0$, these comprise the Temperley–Lieb algebra obeyed by the terms in the n^2-state Potts Hamiltonian. As we already discussed, the operators $H_{i,i+1}$ have the same dimension (n^2) as those in the Potts Hamiltonian. These results may imply that the models are equivalent, with free boundary conditions. However, we do not know a proof of this.

3.2. Four sites

The equivalence can be checked explicitly for the case $L = 2$ (four-site antiferromagnet). The following correspondence of states exists.

Singlets:

$$|{}^{\alpha}{}_{\alpha}{}^{\beta}{}_{\beta}\rangle = n|0, 0\rangle \tag{1a}$$

$$|{}^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha}\rangle = \sum_{\Lambda=1}^{q} |\Lambda, -\Lambda\rangle \tag{1b}$$

(Here we define the H_f eigen-states periodically, $|\Lambda + q\rangle \equiv |\Lambda\rangle$.)

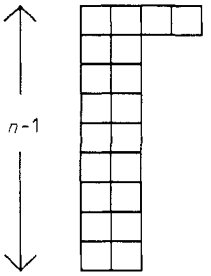

Figure 7. The Young tableau corresponding to the totally traceless states for a chain of four sites.

Adjoint:

$$(T^A)_\alpha{}^\beta|^\alpha{}_\beta{}^\gamma{}_\gamma\rangle = \sqrt{n}|A, 0\rangle \tag{2a}$$

(where $A = 1, \ldots, n^2 - 1$, and the T^As are some orthonormal basis of traceless matrices: tr $T^A = 0$, tr $T^A T^B = \delta^{AB}$.)

$$(T^A)_\alpha{}^\beta|^\gamma{}_\gamma{}^\alpha{}_\beta\rangle = \sqrt{n}|0, A\rangle \tag{2b}$$

$$(T^A)_\alpha{}^\beta|^\alpha{}_\gamma{}^\gamma{}_\beta\rangle = (1/\sqrt{n}) \sum_{\Lambda=0}^{q-1} |\Lambda, A - \Lambda\rangle. \tag{2c}$$

The remaining states are of the form $T_\alpha{}^\beta{}_\gamma{}^\delta|^\alpha{}_\beta{}^\gamma{}_\delta\rangle$, where the tensor **T** is completely traceless with respect to all pairs of upper and lower indices (corresponding to the Young tableau of figure 7). There are $n^4 - 3n^2 + 1$ such states. In Potts language, these are states of the form

$$|A, B\rangle - |B, A\rangle \quad \text{and} \quad |A, A\rangle - |B, B\rangle.$$

There are $q^2 - 3q + 1$ linearly independent states of these types. (It might appear that we should identify

$$|A, B\rangle \propto ???(T^A)_\alpha{}^\beta(T^B)_\gamma{}^\delta|^\alpha{}_\beta{}^\gamma{}_\delta\rangle$$

for some basis of traceless matrices, T^A. However, this is not the case. Instead there is a mixing of the two sites:

$$|C, D\rangle = \sum_{A,B} R^{CD,AB}(T^A)_\alpha{}^\beta(T^B)_\gamma{}^\delta|^\alpha{}_\beta{}^\gamma{}_\delta\rangle$$

for some tensor $R^{CD,AB}$.)

To show the equivalence of these states we just need to show that the antiferromagnetic and Potts Hamiltonians have the same action on the corresponding states.

(The states are not orthogonal, but they are complete. Thus the action of H on the states completely determines the eigenvalues and eigenvectors; i.e. if we have

$$H|\psi_i\rangle = \sum_j h_{ji}|\psi_j\rangle$$

for some complete set of states $|\psi_i\rangle$ in terms of some unsymmetric matrix, h, then an eigenvalue and eigenvector of h

$$\sum_j h_{ij}u_j = \lambda u_i$$

gives $\Sigma_i u_i|\psi_i\rangle$ as an eigenvector of H with eigenvalue λ.)

This equivalence is certainly true for the four-index tensor states; they are all annihilated by H. Thus we just need to consider the singlet and adjoint sectors.

(i) Singlet sector. First note that $|0, 0\rangle$ is an eigen-state of both H_f terms and $\Sigma_{\Lambda=1}{}^q|\Lambda, -\Lambda\rangle$ is an eigen-state of H_P with eigenvalues -1 and $-\lambda$ exactly like $|^{\alpha}{}_{\alpha}{}^{\beta}{}_{\beta}\rangle$ for H_{12}, H_{34} and $|^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha}\rangle$ for H_{23}. Here we used:

$$H_P \sum_{\Lambda=1}^{q} |\Lambda, -\Lambda\rangle = -(\lambda/q) \sum_{\Lambda,\Lambda'} |\Lambda+\Lambda', -\Lambda-\Lambda'\rangle = -\lambda \sum_{\Lambda=1}^{q} |\Lambda, -\Lambda\rangle.$$

The off-diagonal terms are

$$H_f \sum_{\Lambda=1}^{q} |\Lambda, -\Lambda\rangle = -2|0, 0\rangle \qquad H_P|0, 0\rangle = -(\lambda/q) \sum_{\Lambda=1}^{q} |\Lambda, -\Lambda\rangle.$$

For the antiferromagnet we have

$$(H_{12} + H_{34})|^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha}\rangle = -(2/n)|^{\alpha}{}_{\alpha}{}^{\beta}{}_{\beta}\rangle$$
$$H_{23}|^{\alpha}{}_{\alpha}{}^{\beta}{}_{\beta}\rangle = -(\lambda/n)|^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha}\rangle.$$

These equations are equivalent under the identification of states of equations (1), (2)

(ii) Adjoint states: Diagonal terms:

$$H_f|0A\rangle = -|0A\rangle \qquad H_f|A, 0\rangle = -|A, 0\rangle$$
$$H_P \Sigma_\Lambda |\Lambda, A-\Lambda\rangle = -\lambda \sum_{\Lambda} |\Lambda, A-\Lambda\rangle.$$

These are the actions of $(H_{12} + H_{34})$ and H_{23} on the corresponding states. Note that the tracelessness of the tensors T^A implies that the term in H acting on the link contracted with T^A gives 0.

Off-diagonal terms:

$$H_f \sum_{\Lambda} |\Lambda, A-\Lambda\rangle = -|0, A\rangle - |-A, 0\rangle$$
$$H_P|0, A\rangle = H_P|-A, 0\rangle = -(\lambda/q) \sum_{\Lambda} |\Lambda, A-\Lambda\rangle.$$

Again these are the corresponding equations using:

$$H_{12}(T^A)_{\beta}{}^{\alpha}|^{\gamma}{}_{\alpha}{}^{\beta}{}_{\gamma}\rangle = -(1/n)(T^A)_{\beta}{}^{\alpha}\delta^{\gamma}{}_{\alpha}|^{\varepsilon}{}_{\varepsilon}{}^{\beta}{}_{\gamma}\rangle = -(1/n)(T^A)_{\beta}{}^{\gamma}|^{\varepsilon}{}_{\varepsilon}\psi^{\beta}{}_{\gamma}\rangle$$

etc. Thus we have established that all the states are identified as above.

3.3. Arbitrary L in the SU(n) singlet sector

We now consider a chain of arbitrary length. We only consider the SU(n) singlet sector for simplicity but the arguments can be extended to higher representations. A general singlet state corresponds to a valence-bond diagram. Regarding the sites as being paired, $2i-1, 2i$, each pair is either joined by two bonds to other pairs or else is self-contracted. If we follow the paths between connected pairs they must eventually close since they cannot terminate. Thus we obtain closed loops of pairs of sites; a self-contracted pair corresponding to a loop containing only one pair. Let us focus on the set of sites forming a loop. Reordering them, we can write this part of the state as

$$|^{\alpha_1}{}_{\alpha_2}{}^{\alpha_2}{}_{\alpha_3}{}^{\alpha_3}{}_{\alpha_4} \ldots {}^{\alpha_m}{}_{\alpha_1}\rangle.$$

In Potts language, a self-contracted pair corresponds to a state $|0\rangle$, and a longer closed loop of length m to

$$\sum{}' |\Lambda_1, \Lambda_2, \ldots \Lambda_m\rangle$$

where the sum is over all Λ_i subject to the constraint

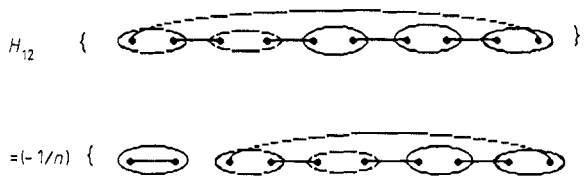

Figure 8. The action of H_{12} on a state containing a closed loop of pairs of sites.

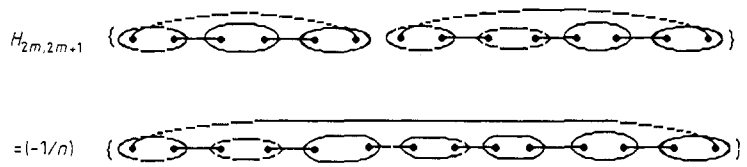

Figure 9. The action of $H_{2m,2m+1}$ on a state with sites $2m$ and $2m+1$ in different closed loops.

$$\sum_{i=1}^{m} \Lambda_i = 0. \tag{3}$$

All SU(n) singlet states have the same squared amplitude, n^L. The Potts states have a squared amplitude $q^{L/2-l}$ where l is the number of loops (since one Λ index is constrained for each loop). Thus, we claim the states are equivalent if the Potts states are multiplied by a factor n^l.

Diagonal terms:

$$H_f|0\rangle = -q|0\rangle$$

$$H_{P12}\sum{}'|\Lambda_1, \Lambda_2, \ldots, \Lambda_m\rangle = -(\lambda/q)\sum_{\Lambda}\sum{}'|\Lambda_1+\Lambda, \Lambda_2-\Lambda, \ldots, \Lambda_m\rangle.$$

We may simply redefine the summation variables: $\Lambda_1 \to \Lambda_1 + \Lambda, \Lambda_2 \to \Lambda_2 - \Lambda$, without interfering with the constraint of equation (3). Thus

$$H_{P12}\sum{}'|\Lambda_1, \Lambda_2, \ldots, \Lambda_m\rangle = -\lambda\sum{}'|\Lambda_1, \Lambda_2, \ldots, \Lambda_m\rangle.$$

Thus we reproduce the effect of H acting on a link containing a valence bond, regardless of whether the link is of $2i-1, 2i$ type or $2i, 2i+1$ type. The off-diagonal terms are

$$H_{f1}\sum{}'|\Lambda_1, \Lambda_2, \ldots, \Lambda_m\rangle = -\sum{}'|0, \Lambda_2, \ldots, \Lambda_m>$$

where now the sum is restricted by

$$\sum_{i=2}^{m} \Lambda_i = 0$$

This corresponds to the valence-bond transition diagram of figure 8. We obtain the correct transition amplitude of $1/n$ due to the difference in normalisation of the two Potts states:

$$H_{Pm,m+1} \sum_{\sum_{i=1}^{m}\Lambda_i=0} \sum_{\sum_{i=m+1}^{m+n}\Lambda_i=0} |\Lambda_1\Lambda_2, \ldots, \Lambda_m, \Lambda_{m+1}, \Lambda_{m+2}, \ldots, \Lambda_{m+n}\rangle$$

$$= -(\lambda/q) \sum_{\sum_{i=1}^{m+n}\Lambda_i=0} |\Lambda_1\Lambda_2, \ldots, \Lambda_m, \Lambda_{m+1}, \Lambda_{m+2}, \ldots, \Lambda_{m+n}\rangle.$$

This corresponds to the valence-bond transitions of figure 9.

Thus we have proven rigorously the equivalence of the two Hamiltonians in the singlet sector. That is sufficient to establish most of the conclusions discussed above about the SU(n) antiferromagnets, since the ground states are expected to be singlets, and for fractional n we only defined the model in the singlet sector.

These models were very recently shown [20] to be equivalent to certain integrable n-state vertex models and results were presented on the correlation length and gap.

Acknowledgment

This research was supported by NSERC of Canada.

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